CLASSIFICATION AND IMPORTANCE OF BARRIER OPTIONS

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Abstract: *In the present paper we explore pricing of continuous and discrete barrier options that become more and more popular because of lower costs than their plain vanilla counterparts. We briefly describe different kinds of barrier options and their importance in Finance. Our financial analysis and the respective mathematical formulation give the possibility to be invented efficient computational methods for valuation of path-dependent derivatives that have not only a simpler computer implementation but also differ with minimum memory requirements and extreme short computational times.*
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1. Introduction

Options are generally defined as a contract between two parties in which one party has the right but not the obligation to do something in a future moment, usually to buy or sell shares of some commodity or stock. The essential of options is that having rights without obligations has financial value because this deal has *no downside* (risk) for the holder, i.e. he either makes a profit or quits without a loss. Thus, he has a potential gain, if he manage to foresee the rising or falling values of the underlying asset buying shares, or otherwise he does not exercise this option. To enter a such contract without obligations, the holder should pay some money in advance, i.e. the *option price* or often known as *premium*.

In the market of financial derivatives the most important problem is the so called *option valuation problem*, i.e. to compute a fair value for the option, i.e. the *premium*. The Nobel Prize-winning Black-Scholes option valuation theory motivates using classical numerical methods for partial differential equations (PDE's) [23]. In computational Finance numerous nonstandard numerical methods are proposed and successfully applied for pricing options [4], [6], [7], [14], [15], [16], [18], [20]. Numerical methods are often preferred to closed-form solutions as it they could me more easily extended or adapted to satisfy all the financial requirements of the option contracts and continuously changing conditions imposed by financial institutions and over-the-counter market for controlling trading of derivatives.

Kunitomo and Ikeda [10] obtained general pricing formulas for European double barrier options with curved barriers but like for a variety of path-dependent options and corporate securities most formulas are obtained for *restricted cases* as *continuous monitoring* or *single barrier* [14]. The *discrete monitoring* is essential as the trading year is considered to consist of 250 working days and a week of 5 days. Thus, taking for one year T=1, the application of barriers occurs with a time increment of 0.004 daily and 0.02 weekly.

For discrete barrier options there are some *analytical solutions*. For example, Fusai reduces the problem of pricing *one barrier option* to a Wiener-Hopf integral equation [7].

Several other different contracts with discrete time monitoring are characterized by updating the initial conditions, such as Parisian options and occupation time derivatives [24]. We remark that although most real contracts specify fixed times for monitoring the asset, academic researchers have focused mainly on *continuous time monitoring models* as the analysis of fixed barriers could be treated mathematically using some techniques such as the *reflection principle* [11]. For example, using the reflection principle in Brownian motions, Li expresses the solution in general as summation of an infinite number of normal distribution functions for standard double barrier options, and in many non-trivial cases the solution consists of *finite terms* [12]. Pelsser derives a formula for continuous double barrier knock-out and knock-in options by inverting analytically the Laplace transform by a contour integration, [22]. Although it could not be claimed that it is impossible to be found an exact or closed-form solution of the Black-Scholes equation [1] for the valuation of discrete double barrier knock-out call option, it is sure that there is a substantial differences in the option prices between continuous and discrete monitoring even for 1 000 000 monitoring dates. This could be trivially tested using correction formula for a single barrier knock-in and knock-out option [3] and for a *double barrier knock-out* call option with the proposed numerical algorithm in [14]. The discrete monitoring considerably complicates the analysis of barrier options [3] and their pricing often requires nonstandard method as those presented in [4], [14], [16], [20].

The advantage of finite difference schemes over close-form formulas is demonstrated in [18], where two different analytical solutions for pricing *continuous* double barrier knock-out call options are analyzed, i.e. the solution is expressed as *infinite series of reflections* and *Fourier series*. For fixed barriers contracts both solutions give the same answer when all the terms have been added up but the main drawback is that the *rate of convergence* of the sum to the solution can be quite different, depending on the *time to expiry*. It is explained how the option valuation problem is substantially complicated by the *presence of two barriers* and their *discrete monitoring*.

The main objective of this paper is to describe different kinds of barrier options and to explain their importance in Finance. Our financial analysis and the respective mathematical formulation give the possibility to be invented efficient computational methods for valuation of path-dependent derivatives that have not only a simpler computer implementation but also differ with minimum memory requirements and extreme short computational times.

Barrier options are traded more than the corresponding standard call and put options owing to the lower price, that is due the additional risk of knock-out or knock-in. In other words this means that the underlying security cancel or activates the option by touching barriers before expiry. Barrier options are very attractive derivative tool for making profit as they have many financial attributes such as dividends and rebate payments, and thus some profit could be guaranteed even if the option is canceled.

2. Importance of Barrier Options in Finance

2.1. Preliminary Notes: Definition and Features of Barrier Options

Barrier options differ from plain vanilla contracts in that the asset price can hit some barrier, at any time prior to expiry T. To explain the great interest in barrier options and their valuation let point the main features that characterizes this kind of options:

1. The importance of these also comes from the fact that barriers can be added to all sorts of existing options, and this is one of the reasons why barrier options have become so popular. An important particular case is that of constant barriers.
2. Barrier options are exotic derivatives that traded in over-the-counter market and they are constantly modified to satisfy the specific customers needs.
3. The additional constraint imposed by barriers makes these options *a much cheaper* and attractive product then the standard options. In fact this limits the holder of possible extreme profits but prevents him also from extreme loss.
4. The holder could be compensated by a *rebate payment* if the options is canceled or not activated, i.e. the barrier has been touched or not breached before maturity in case of knock-out or knock-in options, respectively. Rebates are attractive to many traders as this can make the option more attractive to potential purchasers by compensating them for the loss of the option on knock-out or knock-in event that is unpredictable.
5. *A Path-dependent option*, i.e. the determination of the option value depends on the specific asset path during the life of the option.
6. Barriers may be monitored either *continuously* or *discretely*, i.e. at any moment or daily, weekly, monthly, quarterly that is 150, 50, 12 and 4 times in one trading year.

2.2. Mathematical Interpretation of Barriers and Option Pricing

Now we give the corresponding mathematical formulation and interpretation of each of the above financial characteristics for barrier options:

1. If the asset prices is *log-normally distributed (a variable has a log-normal distribution if the natural logarithm of the variable is normally distribute.)* in the time period [0,T] and given its price today $S_0 = S(0)$ a sample paths for the asset price is generated by the following formula

$$S(t+\delta t) = S(t)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\delta t + \sigma\varepsilon\sqrt{\delta t}\right) \quad (1)$$

where δt is a small time interval. Here ε is a random variable drawing from a standard normal distribution, i.e. $\varepsilon \in N(0,1)$, μ is a measure of the average rate growth of the asset price (known also as *drift*), and σ is a number called the volatility, which measures the standard deviation of the returns. Both parameters are constants in the Black-Scholes framework and $\mu = r$ under risk-neutral measure but in nonstandard option pricing models such as *stochastic volatility models* the volatility is a function of the time. Another example is the Vasicek model [26] describing the evolution of the interest rate as a stochastic process [1].

2. Knock-out options can be further complicated in many ways. For example, the position of the knockout boundary may be a function of time. In particular it may only be active for part of the lifetime of the contract. Another complication for out options is to allow a rebate, whereby the holder of the option receives a specified amount R if the barrier is crossed. The same procedures could be applied for knock-in options.

3. Comparing the price of continuous and discretely monitored barrier options with the corresponding vanilla option with same parameters, in the absence of rebates we have

$$V_{continuous} \leq V_{discrete\ barrier} \leq V_{vanilla\ option} \quad (2)$$

For the price V of barrier options with no rebates we note the following facts:

- There is an analytical solution for plain vanilla options, i.e. the famous Black-Scholes formula [2], a closed-form solution for *continuous* double knock-out barrier option [10], [22], but for *discrete* double barriers are either applied finite difference methods or used nonstandard numerical algorithms [4], [14].
- The price of *continuous barrier options* are lower than their *discrete* counterparts but actually the latter are much more traded in real life owing to the 250 working days in the derivative markets. Discrete valuation is complex task option pricing.
- There is a *substantial* differences in the option prices between continuous and discrete monitoring for 1 000 000 monitoring dates. This could be trivially tested for a single barrier knock-in and knock-out option using formulas [6], [10] or the correction formula [3], for *double barrier* knock-out options with the numerical algorithm [14] or with a high-order accurate finite difference scheme [20].
- The difference is could be negligible only in case when the barrier is *negligible*, i.e. it does not influence on the option price. It is somehow 'invisible' as observed experimentally by Milev and Tagliani [14] and theoretically explained by Ndgomo who proposes a concept of *'critical asset prices'* and postulates theorems for degeneration of double and single barriers to plain vanilla options [21].
- The calculation of the option price for a big number of monitoring dates requires considerable memory resources when finite difference schemes are applied and hence more *computing time*. This is particularly observed for *implicit* schemes and the well-known Crank-Nicolson method that is most frequently used in computational Finance. Avoiding this time obstacle problem requires inventing *nonstandard* finite difference schemes such as that of Milev [18]. The advantage of this scheme is that it is *explicit* and *time-efficient.*

3. Classification of Barrier Options

All over the world numerous types of barrier options are traded every day of the year. They differ according to their financial features (parameters), i.e. *maturity* (fixed or infinite), *exercise features* (early or final), *strike price* (constant or fixed constant but changing with time), *dividends* (continuously compounded or discrete), *volatility* and *interest rate* (constant or depending of time) and *underlying asset price* (log-normally, log-uniform, log-double-uniformly or log-double exponentially (Kou [9]) and normal-inverse Gaussian distributed process (NIG) (Barndorff-Nielsen [1]). In the brackets we have noted how the financial meaning is reflected to the respective parameter in mathematical models. We expose the main types of options in practice that have a *barrier* 'attributes' in their payoff definition (i.e.

the actual initial condition for $\tau = 0$ of the famous Black-Scholes partial differential equation if the time direction t is inverted with the change variable $\tau = T - t$):

1. *Standard single barrier options*, i.e. four basic forms of barrier options depending on the right to exercise that either appears (in) or disappears (out) on *some boundary* in the (S, t) space and whether this happens from above (up) or below (down). The boundary could be a fixed or time-depending barrier such as exponential barriers [11].[3] If the option is canceled, i.e. it becomes zero, but the holder may be compensated by a rebate payment (some traders make profit of barrier options expecting a rebate).
2. *Double barrier options* is defined as the single barrier options but there two barriers, that do not cross each other till maturity. We note that although the definition of single and double barrier options differs in the number of barriers, there is a subtle difference in the valuation of these options in case of *discrete monitoring*. First, there is a correction formula for valuation of *discrete single* barriers option [3] but there is not still proposed a similar closed-form solution for discrete double options. Accurate numerical algorithms with a simple computer implementation are described in [14]. Second, as we mentioned above there are some drawbacks in calculating and extending classical analytical solutions to value such options even in continuous case [10], and sophisticated techniques such as that proposed by Duan et. al. [4] should be applied.
3. *Digital options*, whose payout is fixed after the underlying stock exceeds the predetermined barrier or strike price. Depending on the moment when this happen: if during all the maturity - one-touch options barrier options with a rebate but no other payoff, if only at maturity - cash-or-nothing. A digital put option is explored in [18].
4. *Asset-or-nothing put option* has a payoff that equals to the asset's price if the asset is below the strike price, otherwise the payoff is zero. This options are similar to the above mentioned cash-or-nothing but depends on the *underlying asset at any moment*.
5. *Forward-start barrier options* having a forward-start condition, i.e. at time $t = 0$, when the contract is initiated, it is agreed that at an intermediate time T_1 the holder will receive a barrier contract with later expiry $T > T_1$. The strike and barrier are not known at the outset, but are set by reference to the asset value at T_1, when the barrier option comes into being. A typical example might be a six-month down-and-out call [4].
6. *Perpetual barrier options*, i.e. options with no expiry date. It does not matter when the barrier is reached, only if the asset rises or falls to it, i.e. a digital call or put.

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[3] For example, a down-and-out call option at expiry pays the usual call payoff $\max(S - E, 0)$, provided that S has not fallen from above to the barrier during the life of the option. If S ever reaches the barrier then the option becomes worthless. Obviously the down-and-out call should cost less than the corresponding call, because of the additional risk of knock-out, with premature loss of the premium, i.e. the option price.

[4] Option, starting in three months, with the strike set at-the-money (i.e. at the spot price in three months from initiation) and the barrier set at 90% of this value. Here T_1 is three months and T is nine months. There is no barrier for the first three months of the option's life, and for the next six months the option is a regular down-and-out call.

7. *Ladder options*, if the barrier is triggered, the holder receives (exchanges into) a different contract. This kind of options is very attractive as it locks-in gains once the underlying reaches predetermined price levels or 'rungs', and guaranteeing some profit even if the underlying security falls back below these levels before the option expires.

8. *Barrier options* could be classified according to the information about the underlying asset price and their valuation is important for trading in liquid markets [27].

Of course, this list of barrier options could not be complete as every year derivatives markets expand and numerous new kind of options are designed.

The barrier option has doubled every year since 1992. As we have written above according to the *application of barriers*, i.e. the way the barrier events happen, options could be classification in two main classes, but we will point how this reflects the payoff function from a mathematical point of view:

1. *continuously monitored*, i.e. at any moment the barrier exists and this barrier constraint is a constant or *continuous function* of time;

2. *discretely monitored* daily, weekly, quarterly or monthly. Such options have a *discontinuous payoff condition* and the discontinuity is renewed at every monitoring date.

The mathematical analysis of the first class of options is much more explored than the discrete one as *continuous functions* are handled. Such options have a *discontinuous payoff condition* and usually they are *numerically* valuated in contrast to the continuously monitored ones that has exact option pricing formulas [5]. As a third possibility of single up or down barrier options, i.e. to have more than one barrier, the double knock-out option has both upper and lower barriers where it expires lifeless.

3.1. Valuation of Barrier Options in the Black-Scholes Framework

Usually in financial literature, as a mathematical model for the movement of the asset price S_t under risk-neutral measure is considered a standard *geometric Brownian motion* diffusion process with constant coefficients r (interest rate), q (constant dividend) and σ (volatility), i.e. S_t satisfies the following stochastic differential equation:

$$dS_t / S_t = (r - q)\, dt + \sigma\, dW_t \tag{3}$$

where r - interest rate, σ - stock volatility, dW_t increments of Gauss-Wiener process. By the fundamental Itô's lemma [21] for stochastic equations, the following linear parabolic partial differential equation with non-constant coefficients is derived

$$\frac{\partial V}{\partial t} + (r - q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{4}$$

where t is the current time and the value of the option price $V(S,t)$ at time t before the expiry T is specified as a *solution of this equation* according to the *boundary conditions* that are used [11]. Equation (4) has many solutions, corresponding to all the different options contracts that can be defined with S as a underlying variable. It is known as the Black-Scholes equation [2]. An interesting example of boundary conditions for which the Black-Scholes equation has still not been solved *analytically* is the following one:

Definition 3.1 *A discrete double barrier knock-out call option is an option with a payoff condition equal to* $\max(S - K, 0)$ *which expires worthless if before the maturity the* *asset price has fallen outside the barrier corridor* $[L,U]$ *at the prefixed monitoring dates: at these dates the option becomes zero if the asset falls out of the corridor. If one of the barriers is touched by the asset price at the prefixed dates then the option is canceled, i.e. it becomes zero, but the holder may be compensated by a rebate payment.*

The initial (actual payoff) and boundary conditions of the respective Black-Scholes partial differential equation (4) in case of a *discrete double barrier knock-out* call options are:

(5) $$V(S,0) = (S-K)^{+} 1_{[L,U]}(S)$$

(6) $$V(S,t) \to 0 \ as\ S \to 0 \ and\ S \to \infty$$

with updating of the initial condition at the monitoring dates t_i, $i = 1,\dots,F$:

(7) $$V(S,t_i) = V(S,t_i^-)1_{[L,U]}(S), \quad 0 = t_0 < t_1 < \dots < t_F = T$$

where $1_{[L,U]}(x)$ is the indicator function, i.e., $1_{[L,U]} = 1$ if $S \in [L,U]$ and $1_{[L,u]} = 0$ if $S \notin [L,U]$. It should be noted that away from the monitoring dates, the option price can move on the positive real axis interval $[0,+\infty)$. If there is a *rebate payment*, i.e. the holder is compensated with a fixed amount if one of the barriers L or U is touched before maturity, then we have additional conditions at the monitoring dates t_i, $i = 1,\dots,F$:

(8) $$V(L,t_i) = R_L \text{ and } V(U,t_i) = R_U ,\ i = 1,\dots,F$$

4. Conclusions

Nowadays, the trading of options on financial markets is continuously expanding due to the increasing need of hedging and speculating – two of the basic functions of options. Thus, the classification of options is required and it should be done not by their historical origin on the markets but also according to their payoff properties. Another reason confirming the importance of barrier options and the necessity of their classification is the cheaper price than the respective call and put options. In addition, the classification of barrier options contributes to the financial engineering of path-dependent option and frequently they are known as exotics.

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