PRICING DIGITAL OPTIONS AND VALUATION OF THE GREEKS BY SPECIAL FULLY IMPLICIT SCHEMES

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Abstract: *Using classical finite difference schemes often generates numerical drawbacks such as spurious oscillations in the solution of the famous Black-Scholes partial differential equation. We analyze the fully implicit scheme, frequently used numerical method in Finance, that in presence of low volatility arises spurious oscillations. We propose a modification of this scheme so that we guarantee smooth numerical solution, free of spurious oscillations and satisfies the positivity requirement, as it is demanded for the financial solution of the Black-Scholes equation. The method is used, within the strategy suggested by Rannacher, only in few initial time steps in presence of discontinuous initial conditions. As a consequence, although the method is low order accurate, it returns a spurious oscillations free solution. Next, starting from the smooth initial condition obtained, any other family of arbitrary higher order schemes may be used.*

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1. Introduction

In the market of financial derivatives the most important problem is the so called *option valuation problem*, i.e. to compute a fair value for the option. The solution of the Black-Scholes partial differential equation (PDE) determines the option price, respectively according to the used initial conditions, [1]. This equation is frequently approximated using classical finite difference schemes but they have to satisfy simultaneously several properties: *1. higher order rates of convergence, 2. unconditionally stable, 3. spurious oscillations free, if nonsmooth initial conditions are present*. Then it appears hard task to find methods satisfying simultaneously all requirements 1.-3.. A strategy was suggested in [7] by proposing a scheme in which Crank-Nicolson time-stepping is preceded by a finite number of full implicit steps. The rationale is that high frequency error components will be dampened by the fully implicit steps leading to smooth solution. This strategy is shared also in [14]. The expected rate of convergence remains quadratic since only a finite number of implicit steps are taken. Nevertheless, that strategy is based upon the assumption that the fully implicit scheme is free of spurious oscillations even in presence of low values of volatility. As we will prove in next section, whenever the financial parameters of the Black-Scholes model σ and r satisfy the relationship $\sigma^2 < r$ then the fully implicit scheme produces undesired *spurious oscillations*. The latter are a consequence of complex eigenvalues of the associated iteration matrix.

In this paper we propose a modification of fully implicit scheme which, in the first phase of the strategy suggested by Rannacher [7], operates during the first few time steps in presence of a discontinuous initial conditions. We prove that, even under the severe condition $\sigma^2 < r$, the modified fully implicit scheme 1. provides an accurate solution spurious oscillations free, 2. is positivity-preserving and satisfies the discrete maximum principle, 3. it is time and space consuming, being low order accurate. Nevertheless, it is used only in few time steps. The properties 1., 2., 3. of the modified fully implicit scheme are not trivial. Indeed, both Crank-Nicolson and standard fully implicit scheme, under the restrictive condition $\sigma^2 < r$, provide solutions affecting by spurious oscillations regardless of the used time step, as a consequence of complex eigenvalues in the corresponding iteration matrix.

In Section 2 we discuss shortly the Black-Scholes model. In Section 3 we propose a *modification of the fully implicit finite difference scheme* that enables us to solve accurately the examined PDE. An important factor for numerical schemes is the *condition of positivity of the solution* that must be satisfied as a consequence of the financial meaning of the involved PDE.

In Section 4 we explore digital options that are characterized with discontinuities in the payoff and thus having discontinuous initial conditions in the Black-Scholes equation.

We have pointed out the advantages of the modified fully implicit finite difference scheme. In the conclusion, we give some final remarks for our method.

2. Mathematical Model. The Black-Scholes PDE

We consider the well-known Black-Scholes model [1] for the random movement of the asset price under the risk-neutral measure, i.e. a standard *geometric Brownian motion* diffusion process with constant coefficients r and σ, respectively *interest rate* and *volatility*:

(1) $$dS / S = r\,dt + \sigma\, dW_t$$

The contract to be priced is a digital put option. If t is the time to expiry T of the contract, $0 \le t \le T$, the price $V(S,t)$ of the option satisfies the Black-Scholes PDE

(2) $$-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

endowed with initial and boundary conditions:

(3) $$V(0,S) = A \ \textit{for}\ S \le K,\ V(0,S) = 0\ \textit{otherwise}$$

(4) $$V(t,0) = A\exp(-rt),\ V(t,S_{\max}) = 0\ \textit{for large}\ S_{\max}$$

3. Finite Difference Approach. Preliminaries

As usual, in the finite difference approximation the *S*-domain is truncated at the value $S_{\max}$, sufficiently large such that computed values are not appreciably affected by the upper boundary. The computational domain $[0, S_{\max}] \times [0, T]$ is discretized by a uniform mesh with steps ΔS, Δt. Therefore we obtain the nodes S_j and t_n, where $(S_j = j\Delta S, t_n = n\Delta t)$, $j = 0,\dots,M$, $n = 0,\dots,X$ so that $S_{\max} = M\Delta S$, $T = X\Delta t$, X and M integer numbers.

1. The choice of a specific numerical scheme is based on its property of convergence. The requirement rests on the Lax equivalence theorem.

2. The parabolic nature of the Black-Scholes equation ensures, that being the initial condition $V(0,S) = A \ for \ S \le K$, $V(0,S) = 0$ *otherwise square-integrable*, the solution is smooth in the sense that $V(\bullet,t) \in C^{\infty}(R^{+})$, $\forall t \in [0,T]$. Thus rough initial data give rise to smooth solutions in infinitesimal time.

In some cases, as a consequence the solution obeys the following maximum principle:

(5) $$\max_{S\in[0,S_{\max}]} |V(S,t_1)| \ge \max_{S\in[0,S_{\max}]} |V(S,t_1)|, \qquad t_1 \le t_2$$

This inequality means that the maximum value of $V(S,t)$ should not increase as *t* increases.

If that condition is violated then the numerical solution may exhibit *spurious wiggles near sharp gradients*. As a consequence, even though the numerical method converges, it often yields approximate solutions that differ qualitatively from corresponding exact solutions.

3.1. Modification of the fully implicit scheme

The Crank-Nicolson scheme is *highly accurate*, i.e. $O(\Delta S^2, \Delta t^2)$, but it requires a *prohibitively small time step* for large M to work efficiently [6], [10]. Then in this section we will introduce a less accurate scheme, i.e. $O(\Delta S^2, \Delta t)$, which allows us to choose a more acceptable time step. In the meantime, the scheme prevents from undesired spurious oscillations and guarantees a positive solution.

3.1.1. The standard fully implicit scheme

Preliminary the standard fully implicit scheme is considered, leading to a difference equation

(6) $$AV^{n+1} = V^n$$

where

$$A = tridiag\left\{ -\frac{\Delta t}{2}\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 - r\frac{S_j}{\Delta S}\right]; 1+\Delta t\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + r\right]; --\frac{\Delta t}{2}\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + r\frac{S_j}{\Delta S}\right]\right\}$$

Under the (sufficient) condition $\sigma^2 > r$ it follows

- A is a Jacobi matrix, so that theoretical results about its spectrum are available in literature [11];
- A is an M-matrix, so that $A^{-1} > 0$ and $V^{n+1} > 0$;
- $\left\|A^{-1}\right\|_{\infty} \le \frac{1}{1+r\Delta t}$, (Windisch, 1989, [11]);
- $\left\|V^{n+1}\right\|_{\infty} = \left\|A^{-1}V^n\right\|_{\infty} \le \frac{1}{1+\Delta t}\left\|V^n\right\|_{\infty} < \left\|V^n\right\|_{\infty}$ and then the maximum principle (5) is satisfied;
- A is similar to a symmetric tridiagonal matrix ([8], p.96), so that $\lambda_i(A)$ are real;

- From Gerschgorin theorem $\lambda_i(A) \in \left[\frac{1}{1+\Delta t\left[r+(\sigma M)^2 \right]}; \frac{1}{1+r\Delta t} \right] \subset (0,1)$ and then spurious oscillations are avoided.

If the condition $\sigma^2 > r$ is violated then

- A fails to be a Jacobi matrix and theoretical results about its spectrum are not available in literature;
- positivity of the solution is not guaranteed;
- some $\lambda_i(A)$ may become complex regardess of Δt used. As a consequence, spurious oscillations with consequent negative values of V can occur.

Then, when $\sigma^2 < r$ (as occurs in certain regions of the grid for stochastic volatility models) the standard fully implicit scheme may fail and a proper variant of it is required.

Remark 3.1 *Under the condition* $\sigma^2 < r$ *more suitable schemes are the exponentially fitted schemes [2]. These schemes are implicit, uniformly convergent and first order accurate, i.e.* $\mathrm{O}(\Delta S, \Delta t)$. *Nevertheless, under the restictive condition* $\sigma^2 < r$ *they arise numerical diffusion, requiring severe restrictions on* ΔS *step and loosing their peculiarity [13].*

3.1.2. Modification of the fully implicit scheme

The proposed scheme differs from the standard fully implicit one in the discretization of the reaction term $-rV$ in the Black-Scholes equation (2) by a bivariate approximation. Through a standard procedure we have

(7) $$V(S, t+\Delta t) = \left(V_{j+1}^{n+1} + V_{j-1}^{n+1} \right) + (1-2a)V_j^n$$

with discretization error $\mathrm{O}(\Delta S^2, \Delta t)$ and a arbitrary constant to be determined below. The term $\frac{\partial V}{\partial S}$ is discretized through a centered difference, as well as $\frac{\partial^2 V}{\partial S^2}$.

The stencil of the involved nodes of the scheme are displayed on Figure 1.

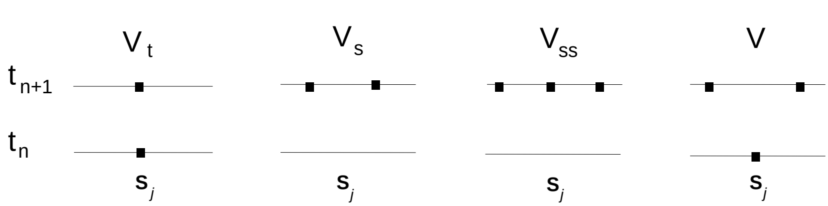

Figure 1: Involved nodes in the fully implicit scheme variant.

The corresponding finite difference equation is

(8)
$$AV^{n+1} = \left[\frac{1}{\Delta t} - r(1-2a)\right]V^{n},$$

where

$$A = tridiag\left\{ ra + \frac{r}{2}\frac{S_j}{\Delta S} - \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 ; \frac{1}{\Delta t} + \left(\frac{\sigma S_j}{\Delta S}\right)^2 ; ra - \frac{r}{2}\frac{S_j}{\Delta S} - \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 \right\}$$

Parameters a and Δt are chosen according to the following criteria:

1. A is an M-matrix, see [11]. Then $ra + \frac{r}{2}\frac{S_j}{\Delta S} - \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 \leq 0$, from which $a \leq -\frac{r}{8\sigma^2}$ (as a consequence $ra - \frac{r}{2}\frac{S_j}{\Delta S} - \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2$ holds);
2. $\frac{1}{\Delta t} - r(1-2a) > 0$, from which $\Delta t < \frac{1}{r(1-2a)}$.

By taking

(9)
$$a = -\frac{r}{8\sigma^2} \qquad and \qquad \Delta t < \frac{1}{r\left(1 + \frac{r}{4\sigma^2}\right)},$$

the following properties follow

- $A = [a_{ij}]$ is row diagonally dominant with $\delta = |a_{ii}| - \sum_{j \neq i} |a_{ij}| = \frac{1}{\Delta t} - \left(\frac{r}{2\sigma}\right)^2 > 0$ and being an *M-matrix* then
  - $A^{-1} > 0$ and from (8) follows $V^{n+1} > 0$, i.e. the proposed modification of the fully implicit scheme makes the scheme positivity-preserving;
  - $\left\|A^{-1}\right\|_\infty \leq \frac{1}{\frac{1}{\Delta t} - \left(\frac{r}{2\sigma}\right)^2}$, (see norm of the inverse of an *M-matrix* in [11]);
- The iteration matrix $A_{iter} = \left[\frac{1}{\Delta t} - r\left(1 + \frac{r}{4\sigma^2}\right)\right]A^{-1}$ has a spectral radius

$$\rho\left(A_{iter}\right) = \left\|\left[\frac{1}{\Delta t} - r\left(1 + \frac{r}{4\sigma^2}\right)\right]A^{-1}\right\|_\infty \leq \frac{\frac{1}{\Delta t} - r - \left(\frac{r}{2\sigma}\right)^2}{\frac{1}{\Delta t} - \left(\frac{r}{2\sigma}\right)^2} < 1$$

so that, under (9), the scheme is *conditionally stable*.

 $\left\|V^{n+1}\right\|_\infty = \left\|A_{iter}V^n\right\|_\infty \le \dfrac{\dfrac{1}{\Delta t} - r - \left(\dfrac{r}{2\sigma}\right)^2}{\dfrac{1}{\Delta t} - \left(\dfrac{r}{2\sigma}\right)^2}\left\|V^n\right\|_\infty < \left\|V^n\right\|_\infty$ so that the solution satisfies *conditionally* the maximum principle (5).

 From Gerschgorin theorem we have

$$\lambda_i\left(A_{iter}\right) \in \left[\frac{\dfrac{1}{\Delta t} - r - \left(\dfrac{r}{2\sigma}\right)^2}{\dfrac{1}{\Delta t} + \left(\dfrac{r}{2\sigma}\right)^2 + 2\left(\sigma M\right)^2};\frac{\dfrac{1}{\Delta t} - r - \left(\dfrac{r}{2\sigma}\right)^2}{\dfrac{1}{\Delta t} - \left(\dfrac{r}{2\sigma}\right)^2}\right] \subset (0,1)$$

Then spurious oscillations are avoided.

The solution accuracy is defined by analyzing the error component introduced by the bivariate approximation of the reaction term $-rV$. The term V_j^{n+1} in the standard fully implicit scheme is replaced by the expression (7) quoted above. By Taylor expansion about the time level $(n+1)\Delta t$ we have

$$a\left(V_{j-1}^{n+1} + V_{j+1}^{n+1}\right) + (1-2a)V_j^n = V_j^{n+1} + \Delta t(-1+2a)\frac{\partial V}{\partial t} + \left(\Delta S\right)\frac{1}{8}^2\left(\frac{r}{\sigma}\right)^2\frac{\partial^2 V}{\partial S^2}$$

Then the scheme is *consistent* with equation (2) with local truncation error $\mathrm{O}(\Delta t) + \mathrm{O}(\Delta S)^2$.

Nevertheless, when $\sigma^2 < r$, both error terms $r\Delta t\left(1 + \dfrac{r}{4\sigma^2}\right)$ and $\left(\Delta S\right)^2\dfrac{1}{8}\left(\dfrac{r}{\sigma}\right)^2\dfrac{\partial^2 V}{\partial S^2}$ become influent and the only constraint (9) provides a poor solution. Then an accurate solution requires

(10) $$\Delta t\left(r + \frac{r^2}{4\sigma^2}\right) << 1 \quad \text{and} \quad \frac{1}{8}\left(\frac{r}{\sigma}\Delta S\right)^2 << 1$$

Then, under (10), the proposed scheme guarantees an accurate solution being positivity-preserving and spurious oscillations free.

4. Numerical Results

In this section we have chosen an example of digital options that have exact *smooth solution* of the Black-Scholes equation. Such options have discontinuity in the payoff function at the strike price K and this produces oscillations in the numerical solution when classical finite difference methods are applied.

The same drawback is observed when the Greek letters Delta (the first partial derivative of option price with respect to asset price) and Gamma (the second derivative) are computed. They are very important quantities to measure the sensitivity of the option prices for hedging. Let us price a digital put option specified by and boundary conditions (3) and (4*).*

The used parameters are A=1, K=10, r=0.05, $\sigma = 0,01$, T=1, S_{max}=2. The analytical solution $V^{ex}(T,S) = A\exp(-rT)(1-N(d_2))$, with $N(\cdot)$ standard normal cumulative distribution and $d_2 = \dfrac{\log(S/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and the numerical solution $V^{imp}(T,S)$ obtained by modified fully implicit scheme presented in section 3.1 are compared in Figure 3.

In the same figure we compare a) the exact Delta, i.e. $\text{Delta}^{ex} = \dfrac{\partial V^{ex}}{\partial S} = -\dfrac{A\exp(-rt)N'(d_2)}{\sigma S\sqrt{T}}$, and an approximate Delta^{imp}, obtained from $V^{imp}(T,S)$ through a centered difference; b) The exact Gamma, i.e. $\text{Gamma}^{ex} = \dfrac{\partial^2 V^{ex}}{\partial S^2} = -\dfrac{A\exp(-rT)d_1 N'(d_2)}{\sigma^2 S^2 T}$, $d_1 = \dfrac{\log(S/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and an approximate Gamma^{imp} obtained from $V^{imp}(T,S)$ through a centered difference scheme.

Here $\sigma^2 < r$ is assumed and standard schemes such as fully implicit and Crank-Nicolson ones fail arising spurious oscillations and negative prices.

The exact V and approximate solution $V^{imp}(T,S)$ are practically indistinguishable. Their difference (and Delta difference) are illustrated in Figure 2 Observing the error, the discontinuity effect is evident in the neighborhood of K. We can note the following advantages of the proposed *modified full implicit scheme* in subsection 3.1.2:

- This implicit scheme is *positivity-preserving* and *oscillations free*.
- The modification of the full implicit scheme works successfully both for the cases $\sigma^2 < r$ and $\sigma^2 > r$, i.e. the accuracy of the scheme is not deteriorated for *low values of the volatility parameter* σ. Pricing *low volatility options* is described by Milev and Tagliani [13].
- The proposed scheme satisfies the *discrete maximum principle (5)*.
- Full implicit schemes could be implemented using a *parallel algorithm* for solving the Black-Scholes equation similar to that proposed by Chiorean [3] or to that in [4].

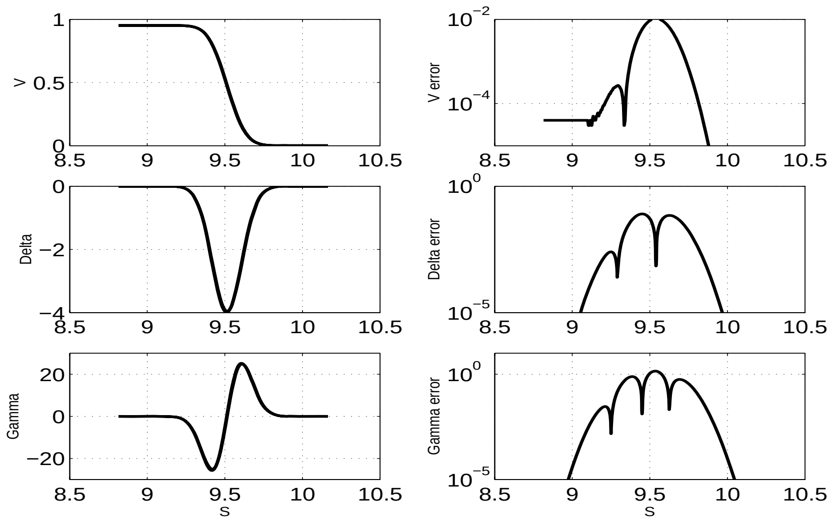

Figure 2: *Pricing* V^{ex}, V^{imp} *and their difference* $\left|V^{ex}-V^{imp}\right|$ *(left and right upper figures); Delta*^{ex}*, Delta*^{imp} *and Delta difference* $\left|Delta^{ex}-Delta^{imp}\right|$ *(left and right median figures); Gamma*^{ex}*, Gamma*^{imp} *and Gamma difference* $|Gamma^{ex} - Gamma^{imp}|$ *(left and right lower). The approximate solution through the modified implicit scheme is done with* $\Delta S = 0.05$ *and* $\Delta t = 0.001$. *The used parameters are: A=1, K=10, r=0.05,* $\sigma = 0.01$, *T=1,* $S_{\max} = 20$.

5. Conclusions

Within the strategy suggested by Rannacher, consisting in a combined use of different finite difference schemes in order to satisfy all the severe requirements of the problem, we have presented an alternative scheme to the classical fully implicit one that do not suffer from spurious oscillations originating from *discontinuous* boundary conditions. This is due to the fact that the scheme has an iteration matrix characterized by real and positive spectrum which allows a fast damping of errors of any order. The proposed *fully implicit scheme* has lower accuracy, i.e. $O(\Delta S^2, \Delta t)$, but it is positivity preserving and satisfies the discrete maximum principle. The scheme operates uniquely during first few time steps, next replaced by an higher order one. The latter starts from a smooth initial condition.

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