ИМПУЛСЕН МОДЕЛ ЗА ДВИЖЕНИЕ НА ЦЕНИТЕ НА СТОКОВИЯ ПАЗАР

Доц. Иванка Стамова, БСУ,
Александър Стамов, Университет на Амастердам

IMPULSIVE MODEL OF PRICE FLUCTUATIONS IN SINGLE COMMODITY MARKETS

Ivanka Stamova, BFU,
Alexander Stamov, University of Amsterdam

Abstract: *In this paper impulsive functional differential equations are proposed to model price shocks in the case of continuous time representation in single commodity markets. The impulses are realized at fixed moments of time. Sufficient conditions for stability of solutions are investigated. The main results are obtained by using the Lyapunov method.*

Key words: Stability, Lyapunov-Razumikhin function, impulsive functional differential price fluctuation model, short-run Economic fluctuations, economic shocks, single commodity market

1. Introduction

First, in a single good market, there are three variables: the quantity demanded q_d, the quantity supplied q_s and its price p. The equilibrium is attained when the excess demand is zero, $q_d - q_s = 0$, that is, the market is cleared. But generally, the market is not in equilibrium and at an initial time t_0 the price p_0 is not at the equilibrium value $\overline{p}$, that is, $p_0 \neq \overline{p}$. In such a situation the variables q_d, q_s and p must change over time and are considered as functions of time. The dynamic question is: given sufficient time, how has the adjustment process $p(t) \to \overline{p}$, as $t \to \infty$ to be described?

The dynamic process of attaining an equilibrium in a single good market model is tentatively described by differential equations, on the basis of considerations on price changes, governing the relative strength of the demand and supply forces. In a first approach and for the sake of simplicity, the rate of price change with respect to time is assumed to be proportional to the excess demand $q_d - q_s$. Moreover, definitive relationships between the market price p of a commodity, the quantity demanded and the quantity supplied are assumed to exist. These relationships are called the demand curve and the supply curve, occasionally modeled by a demand function $q_d = q_d(p)$ or a supply function $q_s = q_s(p)$, both dependant of the price variable p. In the case where the rate of price change with respect to time is assumed to be proportional to the excess demand, the differential equation belongs to the class

(1.1) $$\frac{1}{p}\dot{p}(t) = f(q_d(p), q_s(p))$$

of differential equations. The question that arises is about the nature of the time path $p(t)$, resulting from equation (1.1).

2. The Model Description. Preliminaries

Many authors precisely considered the model (1.1) and its generalizations in order to study the dynamics of the prices, production and consumption for a particular commodity (see [7] and the references cited therein).

In [4] Muresan studied a special case of a fluctuation model for the price with delay $\tau > 0$ of the form

(2.1) $$\dot{p}(t) = \left(\frac{a}{b + p^n(t)} - \frac{cp^m(t-\tau)}{d + p^m(t-\tau)} \right) p(t)$$

where $a, b, c, d, m > 0$, $n \in [1, \infty)$ and proved that there exists a positive, bounded, unique solution.

Rus and Iancu [5] generalized the model (2.1) and studied a model of the form

(2.2) $$\begin{cases} \dot{p}(t) = F(p(t), p(t-\tau))\, p(t), & t > 0, \\ p(t) = \varphi(t), & t \in [-\tau, 0]. \end{cases}$$

They proved the existence and uniqueness of the equilibrium solution of the model considered and established some relations between this solution and coincidence points.

An empirical time series analysis [2] of German macroeconomic data emphasized to model capital intensity, subject to short-term perturbations at certain moments of time. Then it is not reasonable to expect a regular solution of the equation (2.2). Instead, the solution must have some jumps and the jumps follow a specific pattern.

In the long-term planning an adequate mathematical model of this case will be the following impulsive functional differential equation:

(2.3) $$\begin{cases} \dot{p}(t) = F(p(t), p_t)\, p(t), & t \neq t_i,\ t > t_0, \\ \Delta p(t_i) = p(t_i + 0) - p(t_i) = P_i(p(t_i)), & i = 1, 2, \ldots, \end{cases}$$

where $t_0 \in R_+$; $t_0 < t_1 < t_2 < \ldots$, $\lim_{i \to \infty} t_i = \infty$; Ω be a domain in R_+ containing the origin; $F : \Omega \times PC[[-\tau, 0], \Omega] \to R$; $P_i : \Omega \to R$, $i = 1, 2, \ldots$ are functions which characterize the magnitude of the impulse effect at the times t_i; $p(t_i)$ and $p(t_i + 0)$ are respectively the price levels before and after the impulse effects at t_i and for $t > t_0$, $p_t \in PC[[-\tau, 0], \Omega]$ is defined by $p_t(s) = p(t+s)$, $-\tau \le s \le 0$.

We will note that the theory of impulsive differential equation is very well developed [1]. Also, at the present time the qualitative theory of impulsive functional differential equations undergoes rapid development. See, for example [6] and the references therein.

3. Main Results

Let $p_0 \in BC[[-\tau, 0], \Omega]$. Denote by $p(t; t_0, p_0)$, the solution of equation (2.3), satisfying the initial conditions

(3.1) $$\begin{cases} p(t) = p_0(t-t_0), \ t \in [t_0-\tau, t_0], \\ p(t_0+0) = p_0(0), \end{cases}$$

$J^+(t_0, p_0)$ - the maximal interval of type $[t_0, \beta)$ in which the solution $p(t;t_0,p_0)$ is defined, and by $|p_0|_\tau = \max_{t\in[t_0-\tau,t_0]} |p_0(t-t_0)|$ - the norm of the function $p_0 \in BC[[-\tau,0],\Omega]$.

Introduce the following conditions:

H3.1. The function F is continuous on $\Omega \times PC[[-\tau,0],\Omega]$.

H3.2. The function F is locally Lipschitz continuous with respect to its second argument on $\Omega \times PC[[-\tau,0],\Omega]$.

H3.3. There exists a constant $M > 0$ such that

$$|F(p,p_t)| \le M < \infty \quad \text{for } (p,p_t) \in \Omega \times PC[[-\tau,0],\Omega].$$

H3.4. $P_i \in C[\Omega, R]$, $i = 1,2,...$.

H3.5. The functions $(I^1 + P_i): \Omega \to \Omega$, $i = 1,2,...$, where I^1 is the identity in Ω.

H3.6. $t_0 < t_1 < ... < t_i < t_{i+1} < ...$ and $\lim_{i\to\infty} t_i = \infty$

Let $p_1 \in BC[[-\tau,0],\Omega]$. Denote by $p^*(t) = p^*(t;t_0,p_1)$ the solution of equation (2.3), satisfying the initial conditions

$$\begin{cases} p^*(t) = p_1(t-t_0), \ t \in [t_0-\tau, t_0], \\ p^*(t_0+0) = p_1(0). \end{cases}$$

Definition 3.1. The solution $p^*(t)$ of equation (2.3) is said to be:

(a) *stable*, if

$$(\forall t_0 \in R) \ (\forall \varepsilon > 0) \ (\exists \delta = \delta(t_0,\varepsilon) > 0)$$
$$(\forall p_0 \in BC[[-\tau,0],\Omega]: |p_0 - p_1|_\tau < \delta)$$
$$|p(t;t_0,p_0) - p^*(t;t_0,p_1)| < \varepsilon;$$

(b) *uniformly stable*, if the number δ from (a) is independent of $t_0 \in R$;

(c) *attractive*, if

$$(\forall t_0 \in R) \ (\exists \lambda > 0) \ (\forall p_0 \in BC[[-r,0],\Omega]: |p_0 - p_1|_\tau < \lambda)$$
$$\lim_{t\to\infty} p(t;t_0,p_0) = p^*(t;t_0,p_1);$$

(d) *uniformly attractive*, if

$$(\exists \lambda > 0) \ (\forall \varepsilon > 0) \ (\exists T = T(\varepsilon) > 0)$$
$$(\forall p_0 \in BC[[-r,0],\Omega]: |p_0 - p_1|_\tau < \lambda) \ (\forall t_0 \in R)$$
$$(\forall t \ge t_0 + T): \ |p(t;t_0,p_0) - p^*(t;t_0,p_1)| < \varepsilon;$$

(e) *asymptotically stable*, if it is stable and attractive;

(f) *uniformly asymptotically stable*, if it is uniformly stable and uniformly attractive.

Define the class V_0 of all piecewise continuous functions $V: [t_0,\infty) \times \Omega \to R_+$ which are locally Lipschitz continuous with respect to its second argument for $t \ne t_i, i = 1,2,...$, there exist the finite limits

$$\lim_{\substack{t\to t_i \\ t<t_i}} V(t,p) = V(t_i - 0, p), \qquad \lim_{\substack{t\to t_i \\ t>t_i}} V(t,p) = V(t_i + 0, p)$$

and $V(t_i, p) = V(t_i - 0, p)$ for $i = 1,2,\ldots$.

For $t \neq t_i, i = 1,2,\ldots$ and $V \in V_0$ define

$$D^+V(t, p(t)) = \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, p(t+h)) - V(t, p(t))].$$

In our subsequent analysis, we shall use the class V_0 of functions for which:

H3.7. $V(t, p^*(t)) = 0$, $t \geq t_0$

and the following class of functions:

$\Omega_1 = \{p \in PC[[t_0, \infty), \Omega] : V(s, p(s)) \leq V(t, p(t)),\ t - \tau \leq s \leq t,\ t \geq t_0\}$.

Theorem 3.1 *Assume that:*

1. Conditions H3.1-H3.7 hold.

2. There exist functions $V \in V_0$ and $a \in K$ such that

$$a(|p - p^*(t)|) \leq V(t, p), \quad (t, p) \in [t_0, \infty) \times \Omega,$$

$$V(t+0, p + P_i(p)) \leq V(t, p), \quad p \in \Omega_1,\ t = t_i,\ i = 1,2,\ldots.$$

and the inequality

$$D^+V(t, p(t)) \leq 0, \quad t \neq t_i, \quad i = 1,2,\ldots$$

is valid for $t \in [t_0, \infty)$, $p \in \Omega_1$.

Then the solution $p^*(t)$ of equation (2.3) is stable.

Theorem 3.2. *Let the conditions of Theorem 3.1 hold, and a function $b \in K$ exists such that*

$$V(t, p) \leq b(|p - p^*(t)|),\ (t, p) \in [t_0, \infty) \times \Omega.$$

Then the solution $p^*(t)$ of equation (2.3) is uniformly stable.

Theorem 3.3. *Assume that:*

1. Conditions H3.1-H3.7 hold.

2. There exist functions $V \in V_0$ and $a, b, c \in K$ such that

$$a(|p - p^*(t)|) \leq V(t, p) \leq b(|p - p^*(t)|), \quad (t, p) \in [t_0, \infty) \times \Omega,$$

$$V(t+0, p + P_i(p)) \leq V(t, p), \quad p \in \Omega_1,\ t = t_i,\ i = 1,2,\ldots.$$

and the inequality

$$D^+V(t, p(t)) \leq -c(|p(t) - p^*(t)|), \quad t \neq t_i, \quad i = 1,2,\ldots$$

is valid for $t \in [t_0, \infty)$, $p \in \Omega_1$.

Then the solution $p^*(t)$ of equation (2.3) is uniformly asymptotically stable.

4. An Example

Let for a, b, c, $d > 0$, a linear demand function $q_d = a - bp$ and a linear supply function $q_s = -c + dp$ are given and the function $f = \alpha(q_d - q_s)$, $\alpha > 0$. They can be put into (1.1), giving the linear non homogenous differential equation $\frac{1}{p}\dot{p}(t) = \alpha\left(\frac{a+c}{p} - (b+d)\right)$, corresponding to a special type of the differential equation (1.1). Its complementary and particular solutions are immediate.

Mackey and Belair [3] studied the asymptotic behavior of the solutions of the equation

$$\dot{p}(t) = \alpha\left[\frac{a+c}{p(t)} - b - d\,\frac{p(t-\sigma(t))}{p(t)}\right]p(t),$$

where $0 \le \sigma(t) \le \tau$ and τ is a constant.

If at the moments $t_1, t_2, \ldots$ ($t_0 < t_1 < t_2 < \ldots$, $\lim_{i\to\infty} t_i = \infty$) the above equation is subject to impulsive perturbations then the adequate mathematical model is the following impulsive equation

$$(4.1)\qquad \begin{cases} \dot{p}(t) = \alpha\left[\dfrac{a+c}{p(t)} - b - d\,\dfrac{p(t-\sigma(t))}{p(t)}\right]p(t),\ t \ne t_i,\ t > t_0, \\ \Delta p(t_i) = p(t_i+0) - p(t_i) = -\delta_i\left(p(t_i) - \dfrac{a+c}{b+d}\right),\quad i = 1, 2, \ldots, \end{cases}$$

where $t_0 \in R_+$, $p(t)$ represents the price at the moment t, $\delta_i \in R$ are constants, $i = 1, 2, \ldots$.

It is easy to verify that the point $p^* = \dfrac{a+c}{b+d}$ is an equilibrium of (4.1).

We shall show that, if there exists a constant $\beta > 0$ such that $d \le b - \beta$ and the inequalities $0 < \delta_i < 2$ are valid for $i = 1, 2, \ldots$, then the equilibrium p^* of (4.1) is uniformly asymptotically stable.

Let $V(t,p) = \dfrac{1}{2}(p - p^*)^2$. Then the set

$$\Omega_1 = \{p \in PC[[t_0,\infty),(0,\infty)] : (p(s) - p^*)^2 \le (p(t) - p^*)^2,\ t-\tau \le s \le t,\ t \ge t_0\}.$$

For $t > t_0$, $t \ne t_i$ we have

$$D^+V(t,p(t)) = \alpha(p(t) - p^*)[a - bp(t) + c - dp(t-\sigma(t))].$$

Since p^* is an equilibrium of (4.1), then

$$D^+V(t,p(t)) = \alpha\,(p(t) - p^*)[-b(p(t) - p^*) - d(p(t-\sigma(t)) - p^*)].$$

From the last relation for $t > t_0$, $t \ne t_i$ and $p \in \Omega_1$ we obtain the estimate

$$D^+V(t,p(t)) \le \alpha\,[-b+d](p(t) - p^*)^2 \le -\alpha\beta(p(t) - p^*)^2.$$

Also, if $0 < \delta_i < 2$ for all $i = 1, 2, \ldots$, then

$$V(t_i+0, p(t_i+0)) = \frac{1}{2}\left[(1-\delta_i)p(t_i) + \delta_i p^* - p^*\right]^2$$

$$= \frac{1}{2}(1-\delta_i)^2\left[p(t_i) - p^*\right]^2 < V(t_i, p(t_i)).$$

Since all conditions of Theorem 3.3 are satisfied then the equilibrium p^* of (4.1) is uniformly asymptotically stable.

If the constants $\delta_i \in R$ are such that $\delta_i < 0$ or $\delta_i > 2$, then condition 2 of Theorem 3.3 is not satisfied and we can not make any conclusion about the asymptotic stability of the equilibrium p^*.

The example again demonstrates the utility of the second method of Lyapunov. The main characteristic of the method is the introduction of a function, namely, Lyapunov function which defines a generalized distance between $p(t)$ and the equilibrium value p^*.

5. Conclusion

By means of piecewise continuous functions we give the conditions for uniform asymptotic stability of the price p^*. A technique is applied, based on certain minimal subsets of a suitable space of piecewise continuous functions, by the elements of which the derivatives of the piecewise continuous auxiliary functions of Lyapunov are estimated.

It is shown, also, that the role of impulses in changing the behavior of solutions of impulsive models is very important.

References:

1. D. D. Bainov, P. S. Simeonov, Systems with Impulse Effect: Stability, Theory and Applications,Ellis Horwood, Chichester, 1989.

2. J.-F. Emmenegger, I. M. Stamova, Shocks to capital intensity make the Solow equation an impulsive differential equation, International Journal of Differential Equations and Applications, 6 (2002), 93-110.

3. Mackey, M.C. and Belair, J., Consumer memory and price fluctuation in commodity markets: An integrodifferential model, Journal of Dynamics and Differential Equations, 1 (1989), 299-325

4. Muresan, A. S., On some models of price fluctuations in a market economy, Studia Univ. Babes-Bolyai, Math., XXXVIII, 2, (1993),15-19

5. Rus, A. I. and Iancu, C., A functional-differential model for price fluctuations in a single commodity market, Studia Univ. Babes-Bolyai, Math., XXXVIII, 2, (1993), 9-14

6. Stamova, I.M., Stability Analysis of Impulsive Functional Differential Equations, Walter de Gruyter, Berlin, New York, 2009

7. Stamova, I.M. and Emmenegger, G.-F., Stability of the solutions of impulsive functional differential equations modeling price fluctuations in single commodity markets, International Journal of Applied Mathematics, Vol. 15, No. 3 (2004), 271-290